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Answers and Hints

15.2 The Nash Bargaining Model

Let $x = u(\pi_1^o)$ and $y = v(\pi_2^o)$. We maximize $(u(\pi_1) - x)(v(\pi_2) - y)$ subject to $g(\pi_1, \pi_2) = 0$. The solution turns out to depend directly on the second order conditions for constrained maximization. We form the Lagrangian

$$\mathcal{L}(\pi_1, \pi_2, \lambda) = (u(\pi_1) - x)(v(\pi_2) - y) - \lambda g(\pi_1, \pi_2).$$

The first order conditions for this constrained optimization problem are then

$$\begin{aligned} \mathcal{L}_{\pi_1} &= u'(\pi_1)(v(\pi_2) - y) - \lambda g_{\pi_1} = 0 \\ \mathcal{L}_{\pi_2} &= v'(\pi_2)(u(\pi_1) - x) - \lambda g_{\pi_2} = 0. \end{aligned}$$

Now let's differentiate the two first order conditions and the constraint $g(\pi_1, \pi_2) = 0$ totally with respect to player 1's fallback x:

$$\begin{aligned} \mathcal{L}_{\pi_1\pi_1}\frac{d\pi_1}{dx} + \mathcal{L}_{\pi_1\pi_2}\frac{d\pi_2}{dx} + \mathcal{L}_{\pi_1\lambda}\frac{d\lambda}{dx} + \mathcal{L}_{\pi_1 x} &= 0 \\ \mathcal{L}_{\pi_2\pi_1}\frac{d\pi_1}{dx} + \mathcal{L}_{\pi_2\pi_2}\frac{d\pi_2}{dx} + \mathcal{L}_{\pi_2\lambda}\frac{d\lambda}{dx} + \mathcal{L}_{\pi_2 x} &= 0 \\ g_{\pi_1}\frac{d\pi_1}{dx} + g_{\pi_2}\frac{d\pi_2}{dx} &= 0. \end{aligned}$$

Now $\mathcal{L}_{\pi_1 x} = 0, \mathcal{L}_{\pi_2 x} = -v'(\pi_2)$, so we have

$$\begin{bmatrix} \mathcal{L}_{\pi_1\pi_1} & \mathcal{L}_{\pi_1\pi_2} & -g_{\pi_1} \\ \mathcal{L}_{\pi_2\pi_1} & \mathcal{L}_{\pi_2\pi_2} & -g_{\pi_2} \\ -g_{\pi_1} & -g_{\pi_2} & 0 \end{bmatrix} \begin{bmatrix} \frac{d\pi_1}{dx} \\ \frac{d\pi_2}{dx} \\ \frac{d\lambda}{dx} \end{bmatrix} = \begin{bmatrix} 0 \\ v'(\pi_2) \\ 0 \end{bmatrix}$$

Notice that the matrix is the bordered Hessian of the problem, which has positive determinant D by the second order conditions. If we solve for $d\pi_1/dx$ by using Cramer's rule, we get $d\pi_1/dx = g_{\pi_1} g_{\pi_2} v'/D$. The first order conditions imply that g_{π_1} and g_{π_2} have the same sign, so $d\pi_1/dx > 0$.

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15.3 Risk Aversion and the Nash Bargaining Solution

Suppose first that $v(x) = h(u(x))$ where h is concave. Then

$$v'(x) = h'(u(x))u'(x)$$

and

$$v''(x) = h''(u(x))u'^2(x) + h'(u(x))u''(x),$$

so

$$\lambda_v(x) = \lambda_u(x) - \frac{h''(u(x))u'(x)}{h'(u(x))} > \lambda_u(x).$$

Thus $v(x)$ is uniformly more risk averse than $u(x)$.

Now assume $v(x)$ is uniformly more risk averse than $u(x)$. Since u is monotone increasing, u^{-1} exists, so we may define $h(y) = v(u^{-1}(y))$. Clearly $h(0) = 0$. Also $h(y)$ is increasing, since the identity $u(u^{-1}(y)) = y$ implies

$$\frac{d}{dy}u^{-1}(y) = \frac{1}{u'(u^{-1}(y))},$$

and hence

$$\frac{dh}{dy} = \frac{d}{dy}v(u^{-1}(y)) = \frac{v'(u^{-1}(y))}{u'(u^{-1}(y))} > 0.$$

Differentiating again, we get

$$\frac{d^2h}{dy^2} = \frac{\lambda_u(u^{-1}(y)) - \lambda_v(u^{-1}(y))}{v'(u^{-1}(y))u'^2(u^{-1}(y))} < 0,$$

where the last inequality follows from the fact that $v(x)$ is more risk averse than $u(x)$.

15.4 The Ståhl-Rubinstein Bargaining Model with Responent Outside Options

(a) Let π_s be the present value of not bargaining and simply waiting for the outside option to occur. Then we have the recursive equation $\pi_s = ps + (1-p)\delta\pi_s$, which gives the desired result.

(b) From the argument of §5.6, we know the proposed pair of strategies is an SPNE, as long as neither agent has an incentive to seek the outside option. Seeking the outside option for player i is a best response if and only if $ps_i + (1-p)\delta\pi_1 \geq \delta\pi_1$, which reduces to $s_i \geq \delta/(1+\delta)$, which is false by assumption.

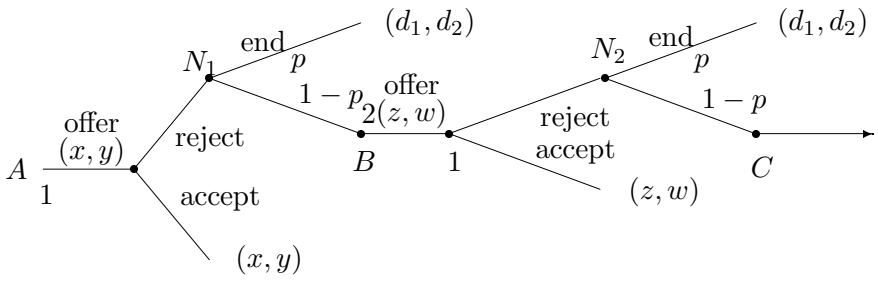
(c) Player 1's strategy as responent is to reject an offer if and only if it is less than $\delta\pi_1$ and as proposer, to offer $1 - \pi_1$. Suppose player 2 seeks the outside option at A'. Then at B, player 2 must offer $\delta\pi_1$, and hence gets $1 - \delta\pi_1$. The value of this at A is $\delta(1 - \delta\pi_1)$. Thus the expected payoff to player 2 at A' is $ps_2 + (1-p)\delta(1-\delta\pi_1)$. It is easy to check that this is just $1 - \pi_1$, so player 2 does no better by deviating in this manner. The only other possibility is that player 2 rejects at A'. This is better than seeking the outside option only if $s_2 > \delta(1 - \delta\pi_1)$. However it is easy to check that this inequality is equivalent to $\delta > (1 + \delta)s_2$, which is false by assumption. This proves that player 2 cannot improve with a single deviation. Can player 1 improve by deviating? Clearly he can gain by offering more at A only if he takes the outside option at B'. But the assumption $s_1 < \delta\pi_1$ means that this is not profitable.

(d) The strategies and payoffs starting at B are the same as in the previous part, with subscripts 1 and 2 interchanged. Therefore player 1 must offer $\delta\pi^*$ and hence receives $\pi_1 = 1 - \delta\pi^*$, where $\pi^* = (1 - (1 - p)\delta - ps_1)/(1 - (1 - p)\delta^2)$. This reduces to the desired expression.

(e) Given the inequalities on s_1 and s_2, it is clear that seeking the outside option is better than rejecting. Can player 2 do better by seeking the outside offer? At C player 1 gets π_1, so he must be offered $ps_1 + (1-p)\delta\pi_1$ at B, so player 2 gets $1 - ps_1 - (1-p)\delta\pi_1$ at B. The value of this at A' is thus $ps_2 + (1-p)\delta(1 - ps_1 - (1-p)\delta\pi_1)$, which must be greater than $1 - \pi_1$. But it is equal to π_1, as simple algebra shows, so player 2 cannot gain by deviating. This also shows that player 1 cannot offer less than $1 - \pi_1$, so neither player can gain by deviating.

15.6 How the Ståhl-Rubinstein Bargaining Model Becomes the Nash Bargaining Model

The game tree is as follows:



Let x be the maximum payoff to agent 1 at C. Then the maximum payoff to agent 1 at N_2 is $pd_1 + (1-p)x$, so agent 2 must offer agent 1 at least $\delta(pd_1 + (1-p)x)$ at B, so the minimum share of agent 2 at B is $g(\delta(pd_1 + (1-p)x))$. Thus the minimum share of agent 2 at N_1 is $(1-p)g(\delta(pd_1 + (1-p)x)) + pd_2$, so agent 1 must offer agent 2 at least

$$\delta(1-p)g(\delta(pd_1 + (1-p)x)) + \delta pd_2$$

at A'. Thus the maximum agent 1 can get at A is

$$g^{-1}(\delta(1-p)g(\delta(pd_1 + (1-p)x)) + \delta pd_2).$$

This must equal x, so we have

$$\delta(1-p)g(\delta(pd_1 + (1-p)x)) + \delta pd_2 = g(x).$$

The solution to this equation is the Ståhl-Rubinstein bargaining equilibrium. If we let $\delta = 1$, this becomes

$$(1-p)g(pd_1 + (1-p)x) + pd_2 = g(x),$$

which we can rewrite as

$$(1-p)\frac{g(x + p(d_1 - x)) - g(x)}{p(d_1 - x)} = \frac{g(x) - d_2}{d_1 - x}.$$

If we take the limit as $p \to 0$, this becomes

$$g(x) = d_2 - g'(x)(x - d_1).$$

But this is just the solution to the Nash bargaining problem

$$\max_x (x - d_1)(g(x) - d_2),$$

as can be seen by differentiating this function and setting it to zero.

15.7 Zeuthen Lotteries and the Nash Bargaining Solution

Suppose the theorem is false, and there are $s, s^* \in S$ such that s^* is a Zeuthen equilibrium, but

$$(u_1(s) - u_1(d))(u_2(s) - u_2(d)) > (u_1(s^*) - u_1(d))(u_2(s^*) - u_2(d)).$$

Then either $u_1(s) > u_1(s^*)$ or $u_2(s) > u_2(s^*)$; assume the former. Let $p^* = (u_1(s^*) - u_1(d))/(u_1(s) - u_1(d))$, so $0 \leq p^* < 1$, and define $p_\epsilon = p^* + \epsilon$ for $0 < \epsilon < 1 - p^*$. Then $p_\epsilon(u_1(s) - u_1(d)) > u_1(s^*) - u_1(d)$ so $p_\epsilon u_1(s) + (1 - p_\epsilon)u_1(d) > u_1(s^*)$, which means that the expected utility from (p_ϵ, s) is greater than the (expected) utility from s; i.e., $(p_\epsilon, s) \succ_1 s$. By assumption, this means that $(p_\epsilon, s^*) \succeq_2 s$. In expected utility terms, this becomes $p_\epsilon u_2(s^*) + (1 - p_\epsilon)u_2(d) \geq u_2(s)$, or $p_\epsilon(u_2(s^*) - u_2(d)) \geq u_2(s) - u_2(d)$. Since this is true for arbitrarily small $\epsilon > 0$, it is true for $\epsilon = 0$, giving $p^*(u_2(s^*) - u_2(d)) \geq u_2(s) - u_2(d)$. But by the definition of p^*, we then have $((u_1(s^*) - u_1(d))/(u_1(s) - u_1(d)))(u_2(s^*) - u_2(d)) \geq u_2(s) - u_2(d)$, or $((u_1(s^*) - u_1(d))(u_2(s^*) - u_2(d)) \geq (u_1(s) - u_1(d))(u_2(s) - u_2(d))$, which contradicts our assumption.

15.8 Bargaining with Fixed Costs

(a) Clearly Mutt cannot do better by deviating. Suppose Jeff deviates on some round and then returns to his prescribed behavior. If any round other than the first is involved, Jeff cannot improve his payoff, because the game is already terminated! To deviate on the first round, Jeff must reject. This costs him c_2. He returns to his prescribed strategy by offering Mutt $1 - c_1$, which Mutt accepts. This leaves Jeff with $c_1 - c_2 < 0$, so Jeff is worse off.

(b) Let l_2 be the least Jeff can get in any subgame perfect Nash equilibrium when he goes first. Since Mutt can get m_1 when it is his turn, Jeff must offer Mutt at most $m_1 - c_1$, which leaves Jeff with at least $l_1 = 1 - m_1 + c_1 \leq 1$. Suppose $l_1 \geq c_2$. The most that Mutt can make is then $1 - (l_1 - c_2)$, so

$$m_1 = 1 - (l_1 - c_2) = m_1 - c_1 + c_2,$$

which implies $c_1 = c_2$, which is false. If $l_1 < c_2$, then Mutt offers Jeff 0, and $m_1 = 1$, which is also a contradiction.

(c) In this case $l_2 = 1$, since Jeff can offer Mutt zero when it is Jeff's turn. But then Mutt can offer Jeff $1 - c_2$, so Mutt earns at least $c_2 > c_1$, which is a contradiction.

15.9 Bargaining with Incomplete Information

(a) The seller chooses p_2 to maximize

$$p\mathrm{P}[v > p] = p\mathrm{P}[\frac{v}{v_2} > \frac{p}{v_2}] = p\left(1 - \frac{p}{v_2}\right),$$

from which the result follows by setting the derivative to zero and solving for p_2;

(b) If $v \geq p_2$, the buyer will accept the second round offer, earning $v - p_2$, which is worth $\delta(v - p_2)$ in the first period. They buyer thus rejects the first offer if and only if $v - p_1 < \delta(v - p_2)$, which gives us the desired condition. If $v < p_2$, p_1 is accepted if and only if $v \geq p_1$.

(c) Suppose $v \geq p_1$ or $v \geq p_2$. If p_1 is rejected, then $v < (p_1 - \delta p_2)/(1 - \delta)$, and since $p_2 = v_2/2$, we can solve for v_2.

(d) Suppose $v \geq p_1$ or $v \geq p_2$. The payoff to the seller from offering p in the first period is

$$p\mathrm{P}[v - p \geq \delta\left(v - \frac{p}{2 - \delta}\right)].$$

This evaluates directly to $p(1 - 2p/v_1(2 - \delta))$, which achieves its maximum when $p = v_1(2 - \delta)/4$. The rest follows by substitution.